Diffusion/Heat Equation

Possible Courses: Linear Algebra, Partial Differential Equations

Application Suppose we have a rod and a given temperature distribution along the rod. Over time, what can we say about the temperature along the rod? Higher dimensional analogues involve modeling the temperature on a (two-dimensional) plate or in a (three-dimensional) room.

Motivated Concepts This module motivates eigenvalues and eigenvectors. This module will also use change of basis to motivate diagonalization. Finally, this application motivates an interpretation of the eigenvalues for a discrete evolution.

Prerequisite Material Students should have already covered the following material in Linear Algebra before beginning the first day of the module: basis (definition and nonuniqueness), coordinates and how the understanding of a vector depends on the current basis, how to create a matrix representation of a known linear transformation, matrix representations are basis dependent, matrix arithmetic, determinants as they are related to the inverse matrix theorem, and change of basis.

Description This project explores an introductory linear algebra approach to solving the one-dimensional heat diffusion problem. We denote the rod's temperature at position x and time t by $u(x,t)$. The goal, then, is to start with an initial temperature distribution $u(x,t_0)$ and find the temperature distribution at a later time t. Figure 1 shows an example of the evolution of one possible initial temperature distribution starting at time $t_0 = 1$ and ending at time $t = 3000$. Students' intuition should allow them to conclude that in the long run, the entire rod should level out to a constant temperature. Indeed, the simulated temperatures in Figure 1 do approach the constant function as time increases. Throughout the module we will leverage this intuition about heat dissipation so that students can verify that the mathematical models reflect the behavior they expect to see.

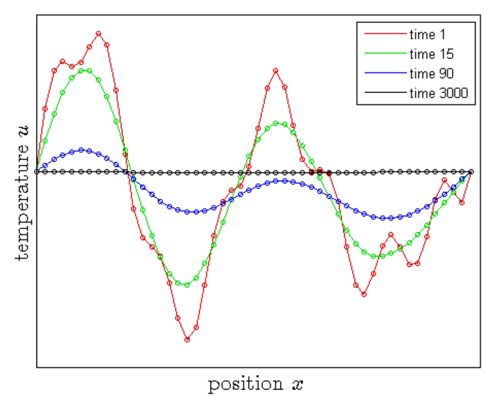

Figure 1: Temperature evolution along a rod at times $t = 1, 15, 90, 3000$; for simplicity we assume the temperature is equal at both ends of the rod and remains constant.

This "diffusion problem" can be modeled with the differential equation

$$\frac{\partial u}{\partial t} = C \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where C is the diffusion coefficient.

Time Frame This module will be 3-4 weeks of a Linear Algebra class. There are a series of homeworks, labs, and class discussions that make up this module and should be implemented as follows:

HW 1 After a class discussion of the application, students are given a homework that asks them to compute successive heat states. The computations are designed to give students the desire for a diagonal matrix. This homework also helps the students get more comfortable with the problem and how matrix multiplication computations can be performed in a less ideal way (without diagonalization).

Lab 1 This lab should happen before any discussion on eigenvalues, eigenvectors, or eigenspaces. This lab is designed to lead students to discover the importance of objects with the properties of eigenvalues, eigenvectors, and eigenspaces ***before students have been formally introduced to them***.

Lab 2 This lab should happen right after students learn about eigenvalues, eigenvectors, or eigenspaces but before diagonalization. The purpose is to motivate diagonalization. Students will gain an appreciation of diagonalization and/or the usefulness of writing vectors as a linear combination of the eigenbasis.

Lab 3 This lab should happen after students learn diagonalization, near the end of the course.(It could also be given as a homework assignment.) This lab is designed to bring together the linear algebra ideas needed for the diffusion model. Students will see that they can be given a scenario that can be described by repeated applications of a linear operator and by knowing the eigenvalues and eigenvectors, they also know the long term behavior of this scenario. Specifically, in the context of this problem, they know that they can write a given heat state as a linear combination of eigenvectors. They discover how knowledge of this linear combination reveals important information about how that state will evolve, and explains some of their observations from the simulations on the first day!

Lab 4 This lab is a culminating activity that asks students to apply what they have learned about 1-dimensional heat diffusion to determine the length of time needed to cool an object that has undergone the process of diffusion welding. The lab can be used as an in-class activity, homework, or writing project.